*Int. J. Solids Structures*, 1971, Vol. 7, pp. 495 to 503. Pergamon Press. Printed in Great Britain

# THE DISCRETE BRANCHING POINT AS A ONE DEGREE OF FREEDOM PHENOMENON

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**Abstract**—The first discrete branching point of a conservative structural system is shown to be essentially a one degree of freedom phenomenon using a generalized coordinates approach. A continuing perturbation scheme which reduces the system to a single degree of freedom is presented and is shown to be valid under a wide range of circumstances. The process is used in conjunction with a topological approach to extend the proofs of two basic theorems of elastic stability to multi-degree of freedom systems with discrete branching points.

## 1. INTRODUCTION

Some insight into the various phenomena which occur in the nonlinear theory of elastic stability has been gained by the discrete representation of continuous systems. The discrete approach was originally introduced in the field of hydrodynamic stability by Poincaré [1] but more recently has been adapted to structures by Thompson [2–5] whose work is comparable to the nonlinear continuum studies of Koiter [6]. In the generalized coordinates approach the emphasis has been placed on branching behaviour at discrete critical points although some considerable thought has been given to snapping behaviour [2] and the coupled buckling behaviour at coincident branching points [7, 8].

It has previously been known that a discrete critical point may be considered as a one degree of freedom problem [1, 6, 9, 10] but there remained a need for this to be authenticated by a process which can be shown to be valid under general circumstances; we need to show that all phenomena associated with the general system are present in the single degree of freedom analysis. To achieve this we introduce a perturbation process which is similar to the type derived previously by Sewell [7, 11, 12] and later fully demonstrated by Thompson and Hunt [13, 14].

The paper secondly focuses attention on two theorems, proved for a single degree of freedom system by Thompson [15], which we may regard as being basic to the field of elastic stability. The first question is posed: can a *fundamental* equilibrium path *thoroughly* [5] lose its stability at a discrete critical point without intersecting a second distinct (*post-buckling*) equilibrium path? We know that the linear approach demonstrates that a critical point is associated with an "adjacent position of equilibrium" but in nonlinear terms we are forced to inquire whether this implies a distinct post-buckling path of *non-zero* length.

The second theorem concerns the stability of the critical point itself. We again consider an initially stable fundamental equilibrium path rising monotonically with the loading parameter but now make no reference to the stability of the post-critical fundamental path, merely insisting that the critical point is unstable owing to nonlinear terms in a Taylor series expansion of the energy function. This may be dangerous because a structural system would snap dynamically from such a point, and furthermore we know that such a system is accompanied by severe imperfection sensitivity. The theorem states that this situation cannot exist without a convenient analytical warning being provided by the approach of a second distinct (*post-buckling*) equilibrium path at sub-critical values of the loading parameter.

## 2. GENERAL THEORY

Let us first consider a discrete conservative structural system described by the total potential energy function $V(Q_i, \Lambda)$ in which $Q_i$ represents a set of $n$ generalized coordinates and $\Lambda$ is a loading parameter. Let us now suppose that in the region of interest the $n$ equilibrium equations $V_i = 0$ yield a single-valued fundamental solution $Q_i = Q_i^F(\Lambda)$, a subscript on $V$ denoting partial differentiation with respect to the appropriate generalized coordinate. We introduce a *sliding* set of incremental coordinates which are defined by

$$Q_i = Q_i^F(\Lambda)+q_i \tag{1}$$

together with a new energy function [3]

$$W(q_i, \Lambda) \equiv V[Q_i^F(\Lambda)+q_i, \Lambda]. \tag{2}$$

Following Thompson [5] we accept as axiomatic the normal equilibrium and stability conditions and note that they hold good for the new energy function.

The quadratic form of the $W$ function may now be diagonalized by means of a non-singular linear transformation of the type $q_i = \alpha_{ij}(\Lambda)u_j$ [3]. Here as elsewhere in the paper unless indicated to the contrary, the repeated suffix summation convention is employed with all summations ranging from 1 to $n$. We shall suppose that one of the infinite number of possible transformations has been chosen, in which $\alpha_{ij}$ is a continuous and single-valued function of $\Lambda$.

It is now possible to introduce the transformed energy function

$$A(u_i, \Lambda) \equiv V[Q_i^F(\Lambda)+\alpha_{ij}(\Lambda)u_j, \Lambda]. \tag{3}$$

The normal equilibrium and stability conditions again hold good for this new energy function which has the properties

$$\begin{aligned} A_i(0, \Lambda) &= A_i'(0, \Lambda) = A_i''(0, \Lambda) = \ldots = 0, \\ A_{ij}(0, \Lambda) &= A_{ij}'(0, \Lambda) = A_{ij}''(0, \Lambda) = \ldots = 0 \quad \text{for } i \neq j, \end{aligned} \tag{4}$$

a subscript again denoting partial differentiation with respect to the corresponding generalized coordinate, and a prime denoting partial differentiation with respect to $\Lambda$. We see we have a valid mapping from the original $\Lambda - Q_i$ space to the new $\Lambda - u_i$ space in which the fundamental path is given by $u_i = 0$.

The diagonalized nature of the transformed energy function presents us with a set of $n$ stability coefficients $A_{ii}(0, \Lambda)$ and we focus attention on a *discrete* critical point on the fundamental path at which a *single* stability coefficient vanishes. Thus it is possible to write

$$\begin{aligned} A_{11}|^c &= A_{11}(0, \Lambda^c) = 0 \\ A_{ss}|^c &= A_{ss}(0, \Lambda^c) \neq 0 \quad \text{for } s \neq 1 \end{aligned} \tag{5}$$

where $\Lambda^c$ is the critical value of $\Lambda$. Having specified a single-valued fundamental equilibrium path, such a critical point will in general correspond to a point of bifurcation.

*The reduction to one degree of freedom*

We now seek to generate a valid scheme of ordered equations which will reduce the problem to one with a single degree of freedom. Let us consider the $n-1$ equilibrium equations

$$A_s(u_i, \Lambda) = 0, \qquad s \neq 1. \tag{6}$$

These equations contain $n+1$ unknowns and therefore the problem may be reduced to one involving *two* independent variables. This enables us to write the generalized coordinates $u_s$ in the parametric form

$$u_s = u_s(u_1, \Lambda) \tag{7}$$

assuming without any loss of generality that the first generalized coordinate and the loading parameter are suitable expansion parameters.

These parametric equations may be substituted into the equilibrium equations $A_s = 0$ to give the identity

$$A_s[u_1, u_t(u_1, \Lambda), \Lambda] \equiv 0, \qquad t \neq 1. \tag{8}$$

Here the left-hand side is a function of the two independent variables so the equations may be differentiated with respect to either or both of these variables as many times as we please. Thus, differentiating repeatedly we generate the ordered equilibrium equations

$$\left.\begin{aligned}
\frac{\partial}{\partial u_1}A_s &= A_{s1} + A_{st}u_{t1} = 0,\\
\frac{\partial}{\partial \Lambda}A_s &= A_{st}u_t' + A_s' = 0,\\
\frac{\partial^2}{\partial u_1^2}A_s &= A_{s11} + 2A_{st1}u_{t1} + A_{stv}u_{t1}u_{v1} + A_{st}u_{t11} = 0,\\
\frac{\partial^2}{\partial u_1 \partial \Lambda}A_s &= A_{st1}u_t' + A_{s1}' + A_{stv}u_{t1}u_v' + A_{st}'u_{t1} + A_{st}u_{t1}' = 0,\\
\frac{\partial^2}{\partial \Lambda^2}A_s &= A_{stv}u_t'u_v' + 2A_{st}'u_t' + A_{st}u_t'' + A_s'' = 0,
\end{aligned}\right\} \tag{9}$$

etc., where $s \neq 1$, $t \neq 1$, $v \neq 1$, etc. As before a prime denotes partial differentiation with respect to the loading parameter and a subscript 1 denotes partial differentiation with respect to $u_1$.

Evaluating the ordered equilibrium equations on the fundamental path and remembering the properties given by (4), these equations may be sequentially solved to give the derivatives

$$\left.\begin{aligned} u_{s1}^F &= u_s'^F = 0, \\ u_{s11}^F &= -\frac{A_{s11}}{A_{ss}}\bigg|^F, \\ u_{s1}'^F &= u_s''^F = 0, \\ u_{s111}^F &= -\frac{1}{A_{ss}}[A_{s111}+3A_{st1}u_{t11}]\bigg|^F, \\ u_{s11}'^F &= -\frac{1}{A_{ss}}[A_{s11}'+A_{ss}'u_{s11}]\bigg|^F \text{ (no summation over } s\text{)}, \\ u_{s1}''^F &= u_s'''^F = 0, \end{aligned}\right\} \tag{10}$$

etc. This continuing process constitutes a valid scheme since the denominator of all derivatives is $A_{ss}^F$ which we have noted is non-zero. Furthermore, one is not compelled to assume a simple critical point in the sense that $A'_{11}|^c \neq 0$ etc. [3].

The evaluation of these derivatives enables $u_s(u_1, \Lambda)$ to be expanded as a power series. This will generate a surface similar to that shown in Fig. 1, which is defined by the $n-1$ conditions $A_s = 0$. Equilibrium paths will lie in this surface and will be further defined by the final condition for equilibrium, $A_1 = 0$. The expressions may be substituted into the $A$ function and we can define a new energy function with only one degree of freedom

$$\mathscr{A}(u_1, \Lambda) \equiv A[u_1, u_s(u_1, \Lambda), \Lambda]. \tag{11}$$

It should be noted that the derivatives of this new energy function may be obtained directly from the known derivatives;

$$\left.\begin{aligned} \mathscr{A}_1 &= A_1 + A_s u_{s1}, \\ \mathscr{A}' &= A_s u_s' + A', \\ \mathscr{A}_{11} &= A_{11} + 2A_{s1}u_{s1} + A_{st}u_{s1}u_{t1} + A_s u_{s11}, \end{aligned}\right\} \tag{12}$$

etc.

*Equilibrium and stability conditions*

Before proceeding further it is necessary to show that the normal equilibrium and stability conditions still hold good for the new energy function. The equilibrium equations $A_s = 0 (s \neq 1)$ have been used in the derivation of the $\mathscr{A}$ function so the equation $A_1 = 0$ is now the necessary and sufficient condition for equilibrium. If we now examine the expression for $\mathscr{A}_1$ given by (12) we find that for equilibrium the second term of the right-hand side vanishes—before evaluation on the fundamental path—by virtue of the original conditions $A_s = 0$. It may therefore be concluded that $\mathscr{A}_1 = 0$ is now both necessary and sufficient for equilibrium and the well known equilibrium condition is preserved within the region of interest.

The stability condition requires a more detailed examination. We know the potential energy $A$ to be stationary with respect to each of the passive coordinates on the curved surface of Fig. 1, and furthermore these stationary points are minima, since we are working

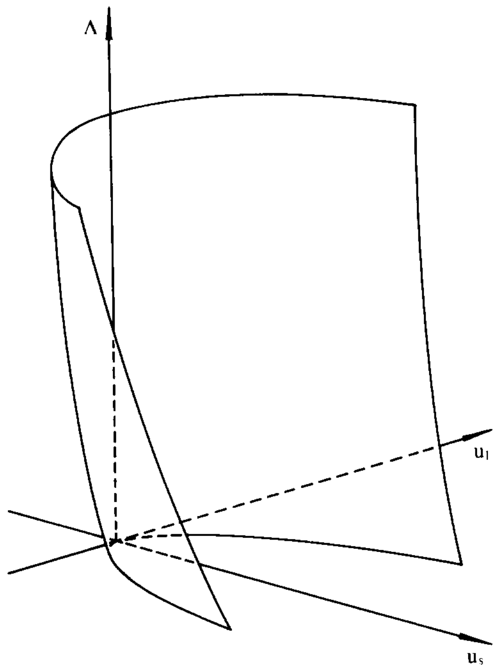


FIG. 1. The surface generated by the $n-1$ equilibrium equations.

in a region of interest which may be thought of as being "stable with respect to the passive coordinates". Some possible A contours in $u_1 - u_s$ space about the stable fundamental path at a fixed Λ level are shown in Fig. 2, in which XX represents the curved surface of interest. YY indicates a section through the A surface at a constant $u_1$ value, and we know this to display a minimum at its intersection with XX. The potential energy surface is therefore

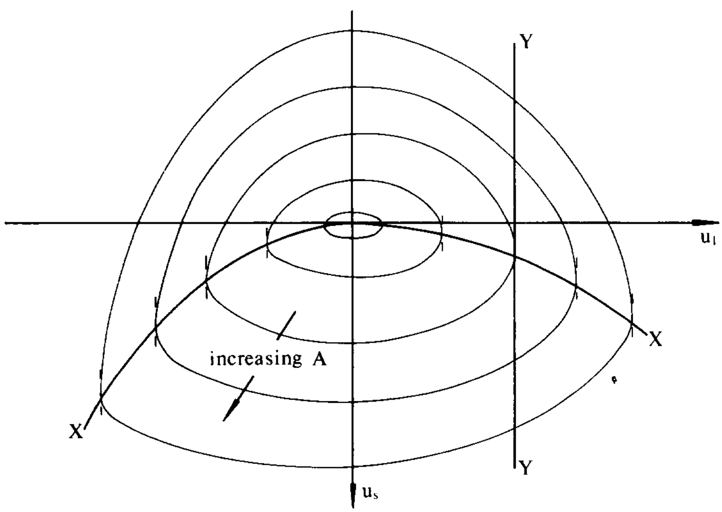


FIG. 2. Energy contours at a fixed load level.

marked with a curve of passive minima—the curve XX—which will pass through all possible equilibrium states at the fixed load level.

Let us now consider the stability of any one of these equilibrium states. If the total potential energy is a minimum along the curve XX, it is certainly a complete minimum; if $A$ increases as we pass out of the equilibrium state along XX, for any other path out of the equilibrium state it will also increase. The $\mathscr{A}$ function operates exclusively along the curve XX, and hence to say that the total potential energy is a minimum along XX is equivalent to saying that $\mathscr{A}$ is a minimum with respect to $u_1$. However, should $\mathscr{A}$ not be a minimum with respect to $u_1$, then the equilibrium state is most certainly unstable since we now have a falling potential energy as we pass along XX. For example, Fig. 3 shows a potential energy surface at a post-critical load level, from which it can be seen that a critical movement away from the unstable equilibrium state would follow the way of passive minima—the curve XX. This reasoning leads us to the conclusion that a minimum of $\mathscr{A}$ with respect to $u_1$ is both necessary and sufficient for stability and the well known stability condition is preserved over the transformation: we note that this conclusion holds for equilibrium states both on and off the fundamental path.

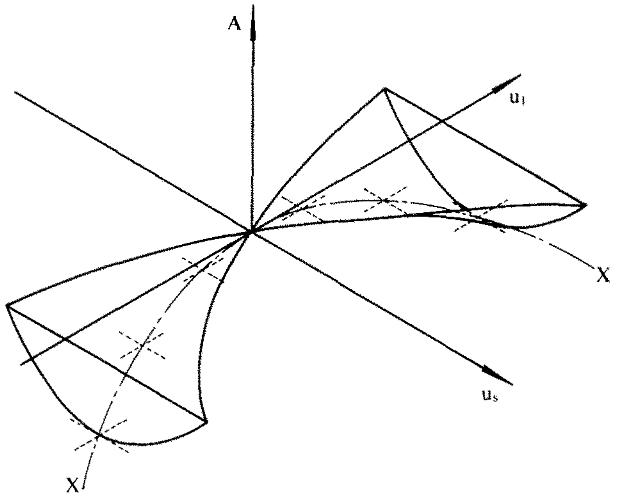


FIG. 3. Potential energy surface at a post-critical load.

Furthermore, if the expression for $\mathscr{A}_{11}$ (12) is examined we find that, with evaluation on the fundamental equilibrium path, all terms but the first vanish by virtue of the equations $A_s = 0$ and subsequent analysis. It is well known that on the fundamental path the stability of the system described by the $A$ function depends initially on the stability coefficient $A_{11}$. This leads to a subsequent conclusion, namely that for the system described by the $\mathscr{A}$ function the necessary and sufficient condition for *thorough* stability [5] along the fundamental path is $\mathscr{A}_{11} > 0$.

## 3. TWO BASIC THEOREMS

The single degree of freedom treatment of a discrete critical point can now be used to prove two basic theorems in elastic stability in greater generality than previously. Thompson

[15] formulates the theorems and presents us with topological and analytical proofs for systems with one degree of freedom. We shall content ourselves here with extending these to a multi-degree of freedom system with a discrete branching point using the powerful topological argument.

Thompson states the first theorem as follows. *An initially-stable (fundamental) equilibrium path rising monotonically with the loading parameter cannot become (thoroughly) unstable without intersecting a second distinct (post-buckling) equilibrium path.* To prove this we first consider the discrete conservative structural system described in the general theory. As has been seen this may then be reduced to the single degree of freedom system described by $\mathscr{A}(u_1, \Lambda)$ providing we are concerned with loss of stability at a discrete branching point. The necessary and sufficient condition for equilibrium is $\mathscr{A}_1 = 0$ and the necessary and sufficient condition for the system to be thoroughly stable on the fundamental path is $\mathscr{A}_{11} > 0$. We have a single-valued *fundamental* equilibrium path which loses its stability at the discrete critical point $(0, \Lambda^c)$ and we investigate the existence of a further *post-buckling* path passing through the critical point.

Consider the variation of the function $F(u_1, \Lambda) = \mathscr{A}_1(u_1, \Lambda)$ within the region of interest. For stable equilibrium $\mathscr{A}$ is a local minimum so we may deduce that in the region surrounding the stable fundamental path the $F$ function adopts the sign of $u_1$. Similarly for a load above the critical load, $\mathscr{A}$ contains a stationary point which is *not* a minimum and $F$ must therefore be negative for positive $u_1$ or positive for negative $u_1$. Normally this stationary point will be a maximum and both conditions hold; this is the situation illustrated in Fig. 4. Assuming the $F$ function to be continuous we find there must be present a *second* equilibrium path defined by $F(u_1, \Lambda) = 0$ which passes through the critical point and the first theorem is proved.

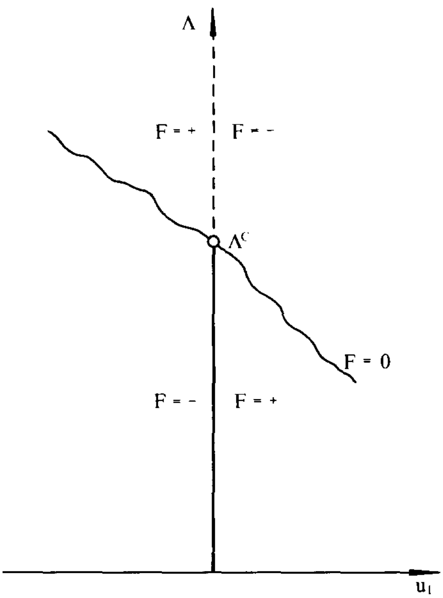


FIG. 4. Topological proof of the first theorem.

The second theorem is concerned with the stability of the critical point itself. Thompson [15] states this as follows. *An initially-stable (fundamental) equilibrium path rising monotonically with the loading parameter cannot approach an unstable critical equilibrium state (from which the system would snap dynamically) without the approach of a second distinct (post-buckling) equilibrium path at sub-critical values of the loading parameter.* As before we consider the discrete conservative structural system of the general theory and reduce it to the single degree of freedom system described by $\mathscr{A}(u_1, \Lambda)$. We again have a single-valued *fundamental* equilibrium path which yields a discrete critical point $(0, \Lambda^c)$ but now make no reference to the stability or instability of the fundamental path above $\Lambda^c$. Within the region of interest the normal equilibrium and stability conditions are preserved over the transformation to a single degree of freedom, despite the fact that at the critical point the stability of the system is not dependent on the second derivative $\mathscr{A}_{11}$ which is now zero, but will rest on higher order terms in a Taylor series expansion.

We again consider the variation of the function $F(u_1, \Lambda) = \mathscr{A}_1(u_1, \Lambda)$ over the region of interest. As before the $F$ function adopts the sign of $u_1$ in the region surrounding the stable fundamental path. If we now suppose the nonlinearities in the Taylor series expansion of the energy function cause the *critical* equilibrium state to be unstable, $F$ must be either negative for positive $u_1$ (as shown in Fig. 5) or positive for negative $u_1$ at $\Lambda = \Lambda^c$. This leads to the conclusion that a *second* equilibrium path defined by $F(u_1, \Lambda) = 0$ which passes through the critical point must approach the fundamental path at sub-critical values of the loading parameter and the second theorem is proved.

## 4. CONCLUDING REMARKS

The single degree of freedom treatment of a multi-degree of freedom system with a discrete branching point has been developed as an ordered perturbation process. The

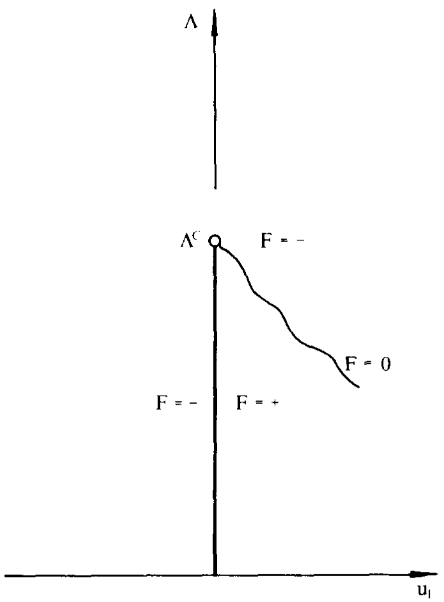


FIG. 5. Topological proof of the second theorem.

continuing scheme does not rely on the common assumption that the critical point is simple, and hence it becomes acceptable over a wide range of circumstances.

The proofs of the basic theorems are limited to the extent that they consider only discrete branching points. A similar approach may be beneficial if we wish to include coincident branching points although this naturally leads to the inclusion of coupled buckling phenomena [7, 8].

*Acknowledgements*—The author would like to acknowledge the generous help he has had from his supervisor Dr. J. M. T. Thompson. This work was conducted under a research studentship from the Science Research Council of Great Britain.

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(*Received* 6 *February* 1970; *revised* 13 *July* 1970)

**Абстракт**—Используя подход обобщенных координат, оказывается, что первая дискретная точка разветвления косервативной системы конструкции представляет собой в основном одну степень свободы. Дается непрерывная схема возмущения, которая сводит систему к системе обладающей одинарной степенью свободы. Эта схема важна для широкого круга обстоятельств. Процесс возмущения используется в месте с тополическим подходом, с целью расширения доказательств двух основных теорем упругой устойчивости к системам с многими степенями свободы и с дискретными точками разветвления.